NON-LINEAR FLIGHT DYNAMICS MODEL OF AN AIRCRAFT

Konstantin Metodiev

Space Research and Technology Institute – Bulgarian Academy of Sciences
e-mail: komet@space.bas.bg

Keywords: *Flight dynamics, quaternion, rotation matrix, Pilatus PC-9M*

Abstract: *In presented report, a nonlinear model of 3D motion of a Pilatus PC-9M aircraft is considered. The system of ordinary differential equations describing the aircraft motion is solved in GNU Octave environment by a 4th-order Dormant-Prince explicit method with an adaptive step, ode45, dopri. Experimental data for stability and control derivative values are used in the model. The developed code is validated with exact solutions. Several 3D maneuvers of the aircraft are simulated by the model. The results are presented graphically and analyzed.*

НЕЛИНЕЕН МОДЕЛ НА ДИНАМИКА НА ПОЛЕТА НА САМОЛЕТ

Константин Методиев

Институт за космически изследвания и технологии – Българска академия на науките
e-mail: komet@space.bas.bg

Ключови думи: *Динамика на полета, кватернион, матрица на ротациите, Pilatus PC-9M*

Резюме: *В настоящия доклад е разгледан нелинеен модел на пространственото движение на самолет Pilatus PC-9M. Системата обикновени диференциални уравнения, описваща движението на самолета, е решена в среда GNU Octave по явен метод Dormant-Prince от 4-ти ред с адаптивна стъпка, ode45, dopri. В модела са използвани експериментални данни за стойностите на производните по устойчивост и управляемост. Разработеният код е валидиран с точни решения. Чрез модела са симулирани няколко пространствени маньовъра на самолета от висшия пилотаж. Резултатите са представени графично и анализирани.*

Introduction

Aircraft non-linear flight dynamics focuses on how aircraft behaves under conditions where the assumptions of linearity (small perturbations and linear relationships between forces and motions) no longer hold. It is crucial for predicting the behavior of aircraft in complex flight conditions. Unlike linear models, nonlinear ones consider full set of equations of motion without simplifying assumptions.

Materials and Methods

Consider following system ODEs for linear and angular momentum conservation (body frame)

(1)
$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}}{m} - \boldsymbol{\omega} \times \mathbf{v} \qquad \frac{d\boldsymbol{\omega}}{dt} = I^{-1}\left(\mathbf{M} - \boldsymbol{\omega} \times I\boldsymbol{\omega}\right) \qquad I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

where **F** = [X, Y, Z]^T, **M** = [L, M, N]^T are vectors of externally applied forces / moments, **v** = [u, v, w]^T, **ω** = [p, q, r]^T stand for linear / angular velocity vectors, *m* is rigid body mass (constant), **I** is inertia tensor describing mass distribution within the rigid body. Vector **ω** is angular velocity vector of a body-fixed reference frame relative to the inertial one.

Euler angular rates [dφ/dt, dθ/dt, dψ/dt]^T might be expressed with regard to body angular rates **ω** = [p, q, r]^T, [1], as follows

(2) $$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \tan\theta\sin\phi & \tan\theta\cos\phi \\ 0 & \cos\phi & -\sin\phi \\ 0 & \dfrac{\sin\phi}{\cos\theta} & -\dfrac{\cos\phi}{\cos\theta} \end{bmatrix} \cdot \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

The dψ/dt equation has a singularity at θ = ± π/2 rad (gimbal lock) which is the reason why quaternion propagation is preferred to propagating Euler angles directly. Quaternion propagation is a method used to describe the object attitude in three-dimensional space over time. The method is more computationally efficient. It involves integrating the quaternion's differential equation with respect to time so as to reflect the airplane changing orientation. Following Shibata, the general form of quaternion propagation equation, neglecting the Earth rate in local frame, is, [2]

(3) $$\frac{dq}{dt} = \frac{1}{2} Q(q)\boldsymbol{\omega} = \frac{1}{2} \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix} \cdot \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

where **ω** = [p, q, r]^T are angular rates in body reference frame, q = [q0, q1, q2, q3]^T are quaternion components. Euler angles yaw ψ, pitch θ, roll ϕ are obtained according to (sequence ZYX only)

(4) $$\begin{aligned} & d = 2\left(q_0 q_2 - q_1 q_3\right), \quad \theta = \operatorname{asin}(d) \\ & if \quad |d| = 1 \quad then \quad \phi = 0, \quad \psi = -2\operatorname{sign}(d)\operatorname{atan}\frac{q_1}{q_0} \\ & else \quad \phi = \operatorname{atan}\frac{2\left(q_0 q_1 + q_2 q_3\right)}{q_0^2 - q_1^2 - q_2^2 + q_3^2}, \quad \psi = \operatorname{atan}\frac{2\left(q_0 q_3 + q_1 q_2\right)}{q_0^2 + q_1^2 - q_2^2 - q_3^2} \end{aligned}$$

Body fixed linear velocities might be transformed into local level reference frame after multiplying by a transformation matrix. The matrix might be derived, according to Kuipers, from quaternions as follows, [3]

(5) $$R(q) = \begin{bmatrix} 2\left(q_0^2 + q_1^2\right) - 1 & 2\left(q_1 q_2 - q_0 q_3\right) & 2\left(q_1 q_3 + q_0 q_2\right) \\ 2\left(q_1 q_2 + q_0 q_3\right) & 2\left(q_0^2 + q_2^2\right) - 1 & 2\left(q_2 q_3 - q_0 q_1\right) \\ 2\left(q_1 q_3 - q_0 q_2\right) & 2\left(q_2 q_3 + q_0 q_1\right) & 2\left(q_0^2 + q_3^2\right) - 1 \end{bmatrix}$$

Aerodynamic coefficients are computed with regard to stability and control derivatives, [1]:

(6)

$$C_x = -C_D \qquad C_l = \frac{\partial C_l}{\partial p}\frac{c}{2V}p + \frac{\partial C_l}{\partial \delta_A}\delta_A$$

$$C_y = \frac{\partial C_y}{\partial \beta}\beta + \frac{\partial C_y}{\partial \delta_R}\delta_R \qquad C_m = C_{m0} + \frac{\partial C_m}{\partial \alpha}\alpha + \frac{\partial C_m}{\partial q}\frac{c}{2V}q + \frac{\partial C_m}{\partial \delta_E}\delta_E$$

$$C_z = -C_{z0} - \frac{\partial C_L}{\partial \alpha}\alpha - \frac{\partial C_y}{\partial \delta_E}\delta_E \qquad C_n = \frac{\partial C_n}{\partial \beta}\beta + \frac{\partial C_n}{\partial r}\frac{c}{2V}r + \frac{\partial C_n}{\partial \delta_R}\delta_R$$

Externally applied aerodynamic forces and moments are computed with the help of widely known formulae including values of aerodynamic coefficients and dynamic pressure. Reference lengths used are mean aerodynamic chord and wingspan. Reference area is wing planform area. The airplane weight is defined in local level reference frame as G = m*[0, 0, –9.81]^T. It has to be transformed to body reference frame by multiplying by transformation matrix (5). In formulae (6), characteristic length c is equal to the mean aerodynamic chord.

Snowden and Keating et al., [4], [5] give experimental data about stability and control derivatives for longitudinal and lateral motion of PC 9/A in cruise configuration, Table 1.

Table 1. Mass parameters, stability and control derivatives (3-2-1-1 maneuver at 5000 ft, maximum likelihood)

Parameter	Value	Units	Parameter	Value	Units	Parameter	Value	Units
Ixx	2505.9	kg.m^2	MAC	1.65	m	Clp	–0.508	1/rad
Iyy	6622.2	kg.m^2	Weight	1866.1	kg	ClδA	–0.1083	1/rad
Izz	8467.1	kg.m^2	CD0	0.0118	–	Cmα	–0.4412	1/rad
Ixy	49.0	kg.m^2	CL0	0.115	–	Cmq	–14.4	1/rad
Ixz	196.9	kg.m^2	CYβ	–0.7735	1/rad	CmdE	–1.2319	1/rad
Iyz	3.0	kg.m^2	CYδR	0.1885	1/rad	Cnβ	0.0808	1/rad
S	16.29	m^2	CLα	5.1222	1/rad	Cnr	–0.201	1/rad
Wingspan	10.125	m	CLδE	0.3151	1/rad	CndR	–0.1157	1/rad

Air density variation with altitude is described by the barometric formula widely available, see for example [6]. However, during simulation, the altitude varies slightly which is why density is assumed constant ρ = 1.293 kg/m^3. This approach simplifies the computational procedure. Lift coefficient at zero angle of attack is assumed CL0 = 0.115. Additional condition for longitudinal static stability is CM0 > 0 i.e., airplane can be trimmed at positive angle of attack. In current case CM0 = 0.01 (lack of data). Also, angle of attack α = atan2(w,u), angle of sideslip β = asin(v / sqrt(u^2 + v^2 + w^2)) From table above and the chain rule we find ∂Cm/∂CL = (∂Cm/∂α) / (∂CL/∂α) ≈ –0.086 times mean aerodynamic chord is the static margin. Savov, Marinov, [7] reported zero lift drag coefficient CD0 = 0.0118 @ M = 0.4; CD0 = 0.0123 @ M = 0.5; CD0 = 0.0127 @ M = 0.6.

After computing external forces and torques, system (1) is integrated in GNU Octave environment by `ode45` solver adopting Dormant – Prince method with adaptive time step.

Validation

Developed source code was put to the test with an exact solution to following problem in rigid body dynamics. Given a projectile is being fired at angle α and initial velocity v0, Fig. 1. The force of gravity G = –mg and the force of air resistance R = –kmv act on the projectile. Find a law describing the projectile motion! Adopted notations and units of measurement are m – mass, kg; v – velocity, m/s; g – Earth gravity acceleration, m/s^2; k – proportional coefficient, 1/s.

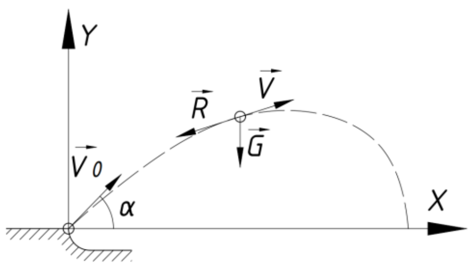

Fig. 1. Validation case according to problem statement

Equations of projectile motion according to Fig. 1 are:

(7) $$m\ddot{x} = -km\dot{x}$$
$$m\ddot{y} = -km\dot{y} - mg$$

These equations are independent of each other, they can be solved separately, and they have the same characteristic equations with real distinct roots, i.e.

(8) $$\rho^2 + k\rho = 0 \quad \rho_1 = 0 \quad \rho_2 = -k$$

Consequently, the general solution takes the form:

(9) $$x = C_1 e^{0t} + C_2 e^{-kt} \quad y = C_3 e^{0t} + C_4 e^{-kt} + u$$

where u = A*t is a particular solution. After replacing the particular integral in second equation of the system (7), we get A = –g/k, i.e., u = –g/k*t. Finally, the complete solution is

(10) $$x = C_1 + C_2 e^{-kt} \quad y = C_3 + C_4 e^{-kt} - gt/k$$

In order to find constants $C_{1..4}$, it is sufficient to plug following initial conditions into system (10):

(11) $$t = 0 \quad x = 0 \quad y = 0 \quad \dot{x} = v_0 \cos\alpha \quad \dot{y} = v_0 \sin\alpha$$

Identical results might be obtained using Symbolic package available in GNU Octave, see Appendix.

Initial conditions (11) were plugged into model being developed, system equations (1) and test problem (7) to perform a simulation of projectile motion and compare obtained solution with exact one. The two results match exactly, Fig. 2.

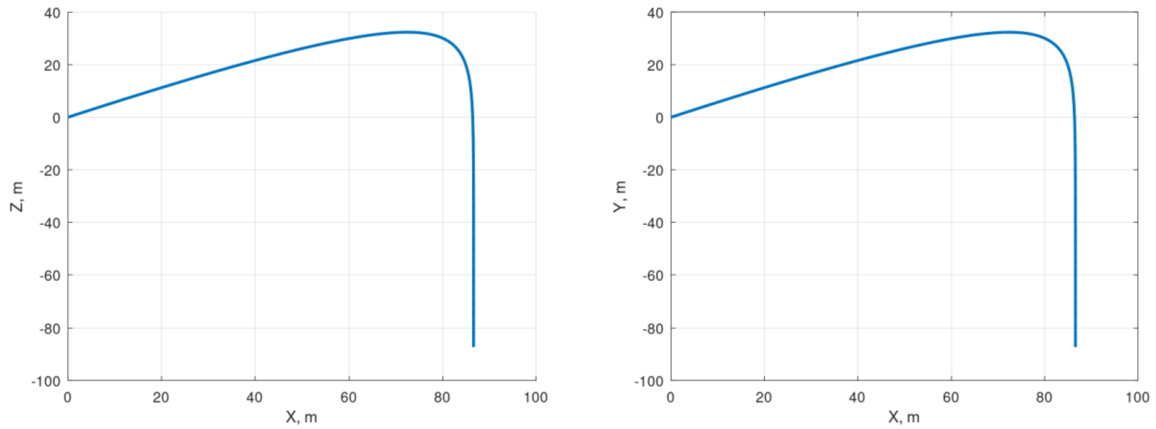

Fig. 2. Validation case results: numerical solution (left) and exact one

In addition, the rotation matrix (5) has been put to the test by MatLab function `quat2rotm(q)` for validation purposes. Again, the obtained results match.

Results

Consider following test cases. PC-9M is performing a flight for 30 seconds at initial (cruise) speed of 140 m/s ≈ 272 knots and yaw angle of –180 deg. The airplane is experiencing thrust (both propeller and jet, [8]) of 6100 N within [0; 30) s interval. A deflection of –2.5 deg is applied to elevator within [5; 30) s interval. Euler angles yaw, pitch, and roll are depicted in Fig. 3, so is the flight path. The figure caption is a ternary condition operator. The airplane is performing a Loop maneuver.

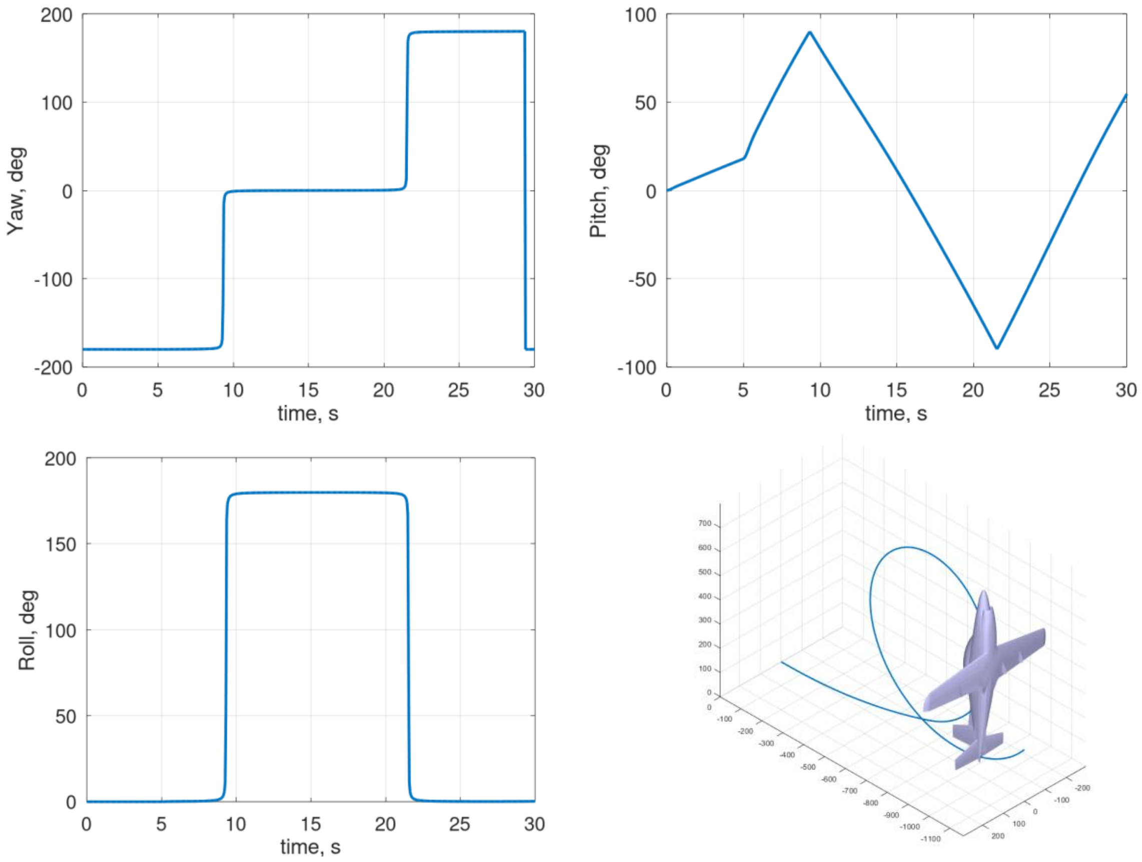

Fig. 3. Loop: dEle = (5 ≤ t < 30 s) ? –2.5 : 0 deg

In following test case a deflection of –1 deg is applied to aileron within [5; 6) s interval, –3 deg to elevator within [6; 15) s to elevator, –2 deg to elevator within [15; 18) s, and +2 deg to aileron within [15; 18) s. The airplane is performing a Chandelle maneuver.

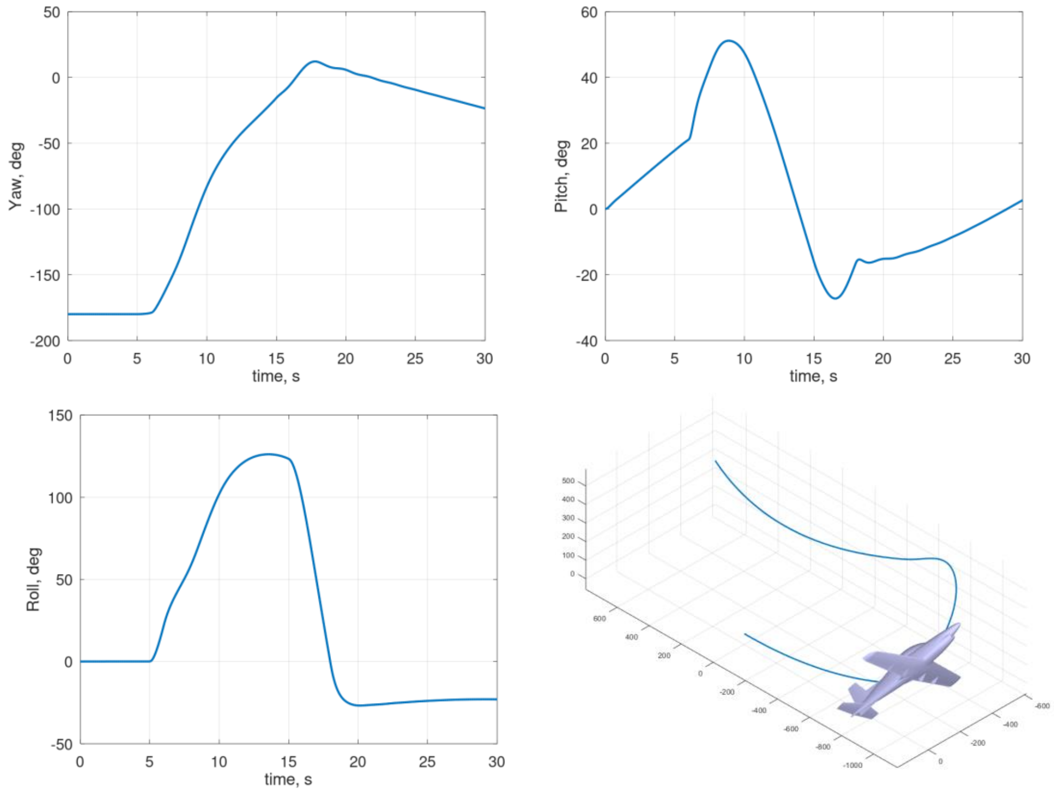

Fig. 4. Chandelle: dAil = (5 ≤ t < 6 s) ? –1 : 0 deg; dEle = (6 ≤ t < 15 s) ? –3 : 0 deg; dEle = (15 ≤ t < 18 s) ? –2 : 0 deg; dAil = (15 ≤ t < 18 s) ? +2 : 0 deg

In following test case, a deflection of –15 deg is applied to elevator within [5; 20) s interval, –2.5 deg to aileron within [0; 20) s interval, thrust is cut to idle, initial (stall) speed of 40 m/s (a recoverable spin).

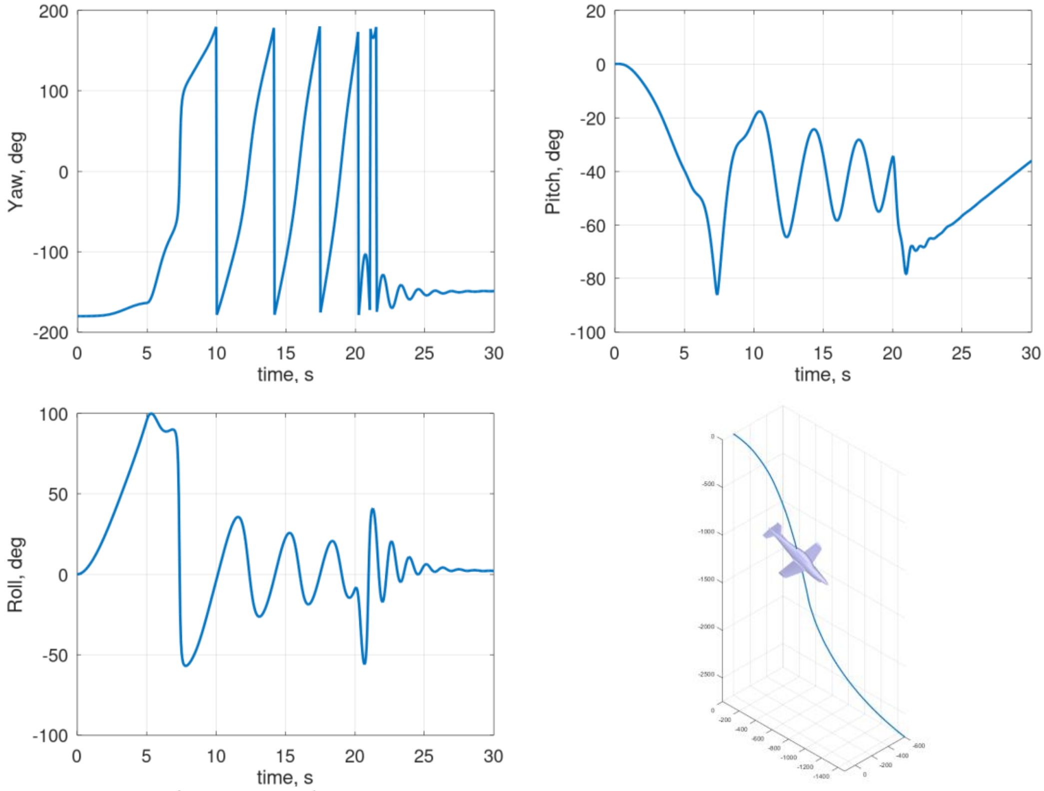

Fig. 5. Spin: dEle = (5 ≤ t < 20 s) ? –15 : 0 deg; dAil = (0 ≤ t < 20 s) ? –2.5 : 0 deg

Conclusion

Data about drag coefficient CD are somewhat rare, so is pitch moment coefficient at zero angle of attack CM0. In paper [7], values of zero lift drag coefficient CD0 at different Mach numbers are reported. Also, in paper [4], Snowden et al. provide experimental data about normal force CN0 @ $\alpha = 0$ (body frame) with regard to values of derivative $\partial CN/\partial q$.

Benefits resulting from applying the non-linear solver are significant. In linearized case, the longitudinal motion is frequently decoupled from longitudinal and lateral in advance and thus various aerodynamic and gyroscopic cross-couplings such as yaw/roll and pitch/yaw are neglected. The outcome of expanding derivatives by Taylor series solely holds within small disturbances about initial conditions. For these reasons, it is possible to compute a three-dimensional motion at critical angles of attack and beyond (recoverable spin, Fig. 5) merely with the aid of a non-linear flight dynamics model.

Project source code was developed in GNU Octave v.9.2.0 environment with free license. The code may be downloaded from github.com, [9].

Obtained numerical results have <u>not</u> been validated during actual flight on PC-9M. Use the results at your sole discretion!

References

1. Josselson, Robert, Introduction to 6-DOF Simulation of Air Vehicles, VisSim Tutorial Series, 1997, Visual Solutions, Inc.
http://avionics.nau.edu.ua/files/doc/VisSim.doc/6dof.pdf
2. Shibata, Minoru, Error Analysis Strapdown Inertial Navigation Using Quaternions, Engineering Notes, AIAA Journal of Guidance, Control, and Dynamics, Vol. 9, No. 3, 1986
3. Kuipers, Jack, Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace and Virtual Reality, Princeton University Press, 1999, ISBN 0-691-10298-8, 125 p.
4. Snowden, A., H. Keating, N. Bronswijk, J. Drobik, A Correlation between Flight-determined Longitudinal Derivatives and Ground-based Data for the Pilatus PC 9/A Training Aircraft in Cruise Configuration, Air Operations Division Aeronautical and Maritime Research Laboratory, DSTO-TR-0937, Commonwealth of Australia, 2000

5. Keating, H., N. Bronswijk, A. Snowden, J. Drobik, A Correlation between Flight-determined Lateral Derivatives and Ground-based Data for the Pilatus PC 9/A Training Aircraft in Cruise Configuration, Air Operations Division Aeronautical and Maritime Research Laboratory, DSTO-TR-0988, Commonwealth of Australia, 2000
6. https://en.wikipedia.org/wiki/Barometric_formula
7. Savov, V., A. Marinov, Aerodynamic Characteristics Calculation of Pilatus-9M Aircraft during Longitudinal and Lateral Motion, Proceedings of Scientific Conference "40 Years since a Man Stepped on the Moon and 30 Years since the First Bulgarian Cosmonaut Flight," pp. 66–71, Vol. 1, Air Force Academy, Dolna Mitropolia, 2009, in Bulgarian
8. Type-Certificate Data Sheet No. EASA IM.E.038 for Engines, PT6A-68 series engines, Pratt and Whitney Canada Corp., April 2016
9. https://github.com/samolet4e/non-linear

Appendix. Source code in GNU Octave for finding exact solution of system (7)

```
>> pkg load symbolic
>> syms x(t) y(t) k g
>> odex = [diff(x,t,2) + k*diff(x,t,1) == 0];
>> odey = [diff(y,t,2) + k*diff(y,t,1) + g == 0];
>> solx = dsolve(odex);
>> soly = dsolve(odey);
>> solx
solx = (sym)
           -k*t
  C1 + C2*e
>> soly
soly = (sym)
           -k*t   g*t
  C1 + C2*e     - ---
                   k
>>
```